Constructions and Obstructions in Birational Geometry

Organizers' report for the ICMS workshop, 5-9th November 2019

March 30, 2019

1 Short description of the meeting

Birational Geometry is about studying similarities and differences between shapes defined by algebraic equations. The simplest such shapes and equations go back to Ancient Greeks, who were concerned with solving the Pythagoras equation $X^2 + Y^2 = Z^2$, where X, Y and Z are positive integers; the first couple of the so-called Pythagorean triples are $(3, 4, 5)$ and $(5, 12, 13)$. In the modern Algebraic Geometric language this problem translates to finding all points (x, y) on a circle $x^2+y^2 = 1$ with coordinates given by rational numbers x and y. These are parametrized by a single rational number t using the formulas

$$(x, y) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right). \tag{1}$$

Nowadays we say that the circle (set of points of a degree two equation in two variables) is **birational** to the line in the t-variable. The ultimate goal of Birational Geometry is to be able to say which algebraic equations define shapes with similar properties. Naturally, there are two ends to work on this problem. One is the **constructions** side which explains how to construct transformations between algebraic equations. One example is the rational parametrization of the circle (1). The other is the **obstructions** side, about why one set of equations can not be transformed into the other. One notable example is the famous Fermat equation

$$x^n + y^n = 1,$$

which does not admit any rational parametrization like (1).

Algebraic Geometry has evolved a lot since antiquity, and is nowadays closely connected with other areas of mathematics, such as Number Theory and Topology as well as to theoretical physics, via String Theory.

In this workshop we invited a variety of experts representing the two approaches, that is constructions and obstructions, to present their work, share ideas and discuss. The special emphasis has been on diversity: we had internationally famous professors, younger lecturers, as well as early career researchers, coming from all over the world. The workshop has been a success, with existing collaborations developed and new collaborations starting. The workshop contributed to existing strengths of the UK Algebraic Geometry community, as well as increased its international visibility.

2 Comprehensive report

The workshop took place between the 5th and 9th November 2018, during the week marking the 114th anniversary of William Edge's birth. There have been a total of 71 participants, out of which

21 were speakers. The Clay Mathematical Institute has subsidized the participation of Prof. Claire Voisin, who was the Clay Institute Lecturer on Nov 6th, as well as four Early Career Researchers: Dr. Barbara Bolognese (University of Sheffield), Lyalya Guseva (Higher School of Economics, Moscow), Enrica Mazzon (Imperial College London) and Dr. Roberto Svaldi (Cambridge University). We are glad to report that they were among the most mathematically active of the participants.

The workshop was centred around two main themes: birational constructions of varieties and obstruction theory. When devising the schedule, we had made a conscious effort to integrate both into each day's schedule in order to allow the audience to access experts in both fields, as it is rare for a mathematician to be interested in a singular approach to their research. We were happy that all scheduled speakers could in fact participate in the conference, which due to delayed arrivals had been unclear at the beginning of the week.

We briefly discuss the scientific content of the workshop, following the structure in the project application. There is a rather transparent caveat - the separation of constructions and obstructions is somewhat a red herring, as in fact they are deeply interlinked - something which became apparent as many talks were cross-referencing from one side to the other throughout the workshop.

2.1 The Constructions side: "Explicit methods in birational geometry" and "Toric degenerations and tropical geometry".

The main objective here is using modern techniques deriving from the Minimal Model Program and moduli spaces, namely Sarkisov links (Kaloghiros, Zimmermann), flips (Casagrande), flops (Brown, Addington) and Mori fibre spaces (Ahmadinezhad) to find new varieties.

A significant portion of the talks has been dedicated their applications, i.e. connecting the above to obtain results in Mirror Symmetry (Corti, Kaloghiros, and Postinghel), Cremona groups (Zimmermann), constructions of Fano varieties (Casagrande) and even philogenetic trees (Maclagan). The speakers have taken great care in providing interesting examples (especially in dimensions three and four) arising both as motivation to their talks but more often as discoveries of new objects (Casagrande, Kaloghiros, Maclagan), or techniques (Brown).

Prof. Borisov presented his recent explicit construction of fake projective planes and related surfaces. This has been an open question since Mumford's first construction of a fake projective plane in 1979, which was done using p-adic uniformization.

In Dr. Pieropan's talk, the minimal model program has been applied to study existence of rational points of rationally connected varieties over C_1 fields, providing a very interesting interplay between birational and arithmetic geometry.

2.2 The Obstructions side: "Cohomological obstructions to stable rationality" and "Derived categories and the Grothendieck ring"

In the key talk of the workshop, Prof. Voisin has given a clear and accessible overview of her recent work on various versions of the decomposition of the diagonal for rationally connected varieties, as an obstruction to rationality.

Prof. Colliot-Thélène presented families of explicit examples of cubics admitting the decomposition of the diagonal, in a talk that proved to be a beautiful mix between constructions and obstructions, in the spirit of the workshop.

Prof. Ottem talked about his recent work on new counterexamples to the integral Hodge conjecture. The proof used a specialization method, reminiscent to Prof. Voisin's.

The talk of Prof. Addington involved birational geometry done via matrix factorizations, which is at the frontier of modern algebraic geometry, with connections to physics.

Prof. Gorchinskiy presented his recent work on classes of varieties with group actions in the Grothendieck ring of varieties, which included the proof of a conjecture by Galkin and Shinder, and counterexamples to more general statements.

Prof. Kuznetsov talked about his new work on derived categories of singular toric surfaces. This talk has perhaps the best possible mix between constructions and obstructions: if the the Brauer group of a toric is nonzero (obstruction), then its derived category does not admit a semiorthogonal decomposition into finite-dimensional algebras, whereas if the obstruction is zero, an explicit construction to decompose the derived category is given.

2.3 Deviation from the original proposal

The Constructions side was slightly unbalanced towards the "Explicit methods in birational geometry" topic as opposed to "Toric degenerations and tropical geometry". We had hoped that there would be a nucleus formed by Prof. Dimca and Dr. Pokora, who have rave rarely met but work with the same objects, however the former cancelled his participation and the latter chose to lean more towards the "Explicit methods" topic. The talks predominantly centred on toric and tropical geometry were by Prof. Maclagan and Prof. Postinghel, though this remains a topic referenced many times in other talks as a good source of motivating examples.

Together with Ms. Wasley, we had discussed whether we would include a public lecture on Thursday evening, and after much debating we resolved not to. Firstly, there was the problem of space: the room was already almost at full capacity every day, therefore at least half of the audience would have been formed of experts in the field. Secondly, it had not been clear to us at the very beginning that the aim of a public lecture was for a very general audience (and not for at least an undergraduate mathematics level). This required finding a completely different speaker than the person we initially had had in mind, and we had some cancellations from the more experienced participants who would have been compatible with the task. We essentially had a choice between Prof. Voisin (who was already Clay Lecturer) and Prof. Kuznetsov, who only managed to obtain his visa the day before his talk was scheduled. Instead, we made an explicit effort to keep the schedule light in order to encourage discussions, which generally lasted until the closing of the Institute.

2.4 Interactions, collaboration and feedback

There have been numerous interactions between the speakers and early career researchers. From the very beginning, the atmosphere was vibrant and the blackboards next to the lecture hall were constantly occupied with intense mathematical discussions. This fact was remarked upon by Ms. Spencer, Ms. Wasley and many participants and is visible during the video recorded on location. Among the very popular speakers were Claire Vosiin and Alexander Kuznetsov. We received very positive feedback from the participants, with the word "excellent" occurring 123 times! This referred both to quality and level of the talks, interactions and overall organization due to ICMS. We'd like to mention that Ms. Spencer, Ms. Walker and Ms. Wasley were of immense help throughout and that we owe them a large part of the success of this workshop.

Among recent existing and developed collaborations between the participants of the workshop are the following articles (with `arXiv.org` identifier):

1. Brown-Fatighenti `arXiv:1707.00653`
2. Brown-Wemyss `arXiv:1707.01150`
3. Brown-Qureshi `arXiv:1707.00736`

4. Battistella-Carocci arXiv:1804.06048, arXiv:1801.07739
5. Cheltsov-Shramov arXiv:1809.09223
6. Das Dores-Mauri arXiv:1808.05023
7. Heden-Zimmermann arXiv:1706.02533
8. Kuznetsov-Shinder arXiv:1809.10628
9. Li-Petrusi arXiv:1802.01134
10. Shinder-Voisin arXiv:1903.02111

This list does not include joint work between supervisors and their (former) PhD students, or collaborations between mathematicians who are currently at the same institution.

Given that mathematical collaborations take time to develop and mature we believe that a lot more joint papers between the participants will appear in the future.

We sincerely thank the ICMS and the Clay Institute for their collaboration and support in organising this event. The workshop had also had significant financial support from Arend Bayer's ERC grant.

Arend Bayer, Ivan Cheltsov, Liana Heuberger, and Evgeny Shinder